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Modified Algo Method

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Abstract

In this paper, we establish a modified new iterative method for solving nonlinear equations extracted from New iterative Method. Modified Algo method has convergence order 2.6180 and efficiency index 1.680 . The proposed method is then applied to solve some problems in order to assess its validity and accuracy.

Keywords: fixed point method, new iterative method, modified algo method

1 Introduction

Solving equations in one variable is the most discussed problem in numerical analysis. Their are several numerical techniques for solving nonlinear equations (see for example [[1]-[3]]). Fixed point iteration method [4] is the fundamental algorithm for solving nonlinear equations in one variable. It has first order convergence. In his method a function f is given and we have to find at least one solution to the equation $f\ (x) = 0$. Note that, priorly, we do not put any restrictions on the function f , we need to be able to evaluate the function, otherwise, we can not even check that a given solution is true, that is, $f(\alpha) = 0$. In reality, the mere ability to be able to evaluate the function does not suffice. We need to assume some kind of “good behavior.” The more we assume, the more potential we have, on the one hand, to develop fast algorithms for finding the root. At the same time, the more we assume, the fewer the functions are going to satisfy our assumptions! This is a fundamental paradigm in numerical analysis. We know that one of the fundamental algorithm of solving nonlinear equations is so called fixed point iteration method. In this method equation is rewritten as

$$x = g(x), \tag{1}$$

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where
(i) there exist $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$,

(ii) there exist $[a, b]$ such that $|g'(x)| \leq L < 1$ for all $x \in [a, b]$
Considering the following iteration scheme:

$$x_{n+1} = g(x_n), n = 0, 1, 2, \cdots \tag{2}$$

and start with a suitable initial approximation x_0, we build up a sequence of approximations, say $\{x_n\}$, for the solution of the nonlinear equation, say
α.The scheme will converge to the root α, provided that
(i) the initial approximation x_0 is chosen in the interval $[a, b]$,
(ii) g has a continuous derivative on(a, b),
(iii) $|g'(x)| < 1$ for all $x \in [a, b]$.
(iv) $a \leq g(x) \leq b$ for all $x \in [a, b]$. (see [4]).
During the last many years, the numerical techniques for solving nonlinear equations have been successfully applied (see, e.g., [2–4] and the references therein). In [4], Babolian and Biazar modified thet there exist $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$ standard Ado- mian decomposition method for solving the nonlinear equation f(x)= 0 to derive a sequence of approximations to the solution, with nearly super-linear convergence. However, their method requires them computation of higher-order derivatives of the nonlinear operator involved in the functional equation.
Kang et al. [14] described a new second order iterative method for solving nonlinear equations extracted from fixed point method by the following approach of [3] as follows: If $g'(x) \neq 1$ can modify by adding $\theta \neq -1$ to both sides as:

$$\theta x + x = \theta x + g(x)$$

$$(1 + \theta)x = \theta x + g(x)$$

which implies that

$$x = \frac{\theta x + g(x)}{1 + \theta} = g_\theta(x) \tag{3}$$

In order for $g_\theta(x)$ to be efficient we choose θ such that $g'_\theta(x) = 0$, we yields

$$\theta = -g'(x)$$

so that (3) take the form

$$x = \frac{-xg'(x) + g(x)}{1 - g'(x)}$$

For a given x_0, we calculate the approximation solution x_{n+1} by iterative scheme

$$x_{n+1} = \frac{g(x_n) - x_n g'(x_n)}{1 - g'(x_n)}, \quad g'(x_n) \neq 0$$

Definition 1 *Let $\{x_n\}$ converge to α with convergence order p. If there exist an integer constant p, and real positive constant C such that*

$$\lim_{n \longrightarrow \infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C \tag{4}$$

Theorem 2 *[3] Suppose that $g \in C^p[a,b]$. If $g^{(k)}(x) = 0$ for $k = 1, 2, 3, \cdots, p-1$ and $g^{(p)}(x) \neq 0$, then the sequence $\{x_n\}$ is of order p.*

It is well known that the fixed point method has first order convergence and New Iteration method has second order convergence.

During the last many years, the numerical techniques for solving nonlinear equations has been successfully applied (see for example [1], [2], [3]).

2 Modified Algo Method

Consider the nonlinear equation

$$f(x) = 0, x \in \mathbb{R} \tag{5}$$

.We assume that α is simple zero of $f(x) = 0$ and x_0 is an initial guess sufficiently close to α.The equation 5 is usually rewritten a $g(x) = x$ and we have a 2nd order iterative scheme [14], whose functional equation is

$$x = \frac{g(x) - xg'(x)}{1 - g'(x)} \tag{6}$$

In this paper, following the approach of McDougall and Wotherspoon [15] and using 6. we propose a simple and rapid convergent method as follows:

Algorithm 3

$$y_0 = x_0$$

;

$$x_1 = \frac{g(x_0) - xg'(x_0)}{1 - g'(x_0)}$$

,

$$y_n = \frac{g(y_{n-1}) - y_{n-1}g'(x_n)}{1 - g'(x_n)} \tag{7}$$

,

$$x_{n+1} = \frac{g(y_n) - y_n g'(x_n)}{1 - g'(x_n)} \tag{8}$$

.The proposed method requires only two evaluations and has convergence order $\frac{3+\sqrt{5}}{2}$.Now we discuss the convergence analysis of Algorithm 3

3 Convergence Analysis

Theorem 4 *Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ for an open interval D and consider that the nonlinear equation $f(x) = 0$(or $g(x) = x$) has a simple root $\alpha \in D$, where $g(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ and $g(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be sufficiently smooth in the neighborhood of the root α; then the order of convergence of Algorithm 3 is $\frac{3+\sqrt{5}}{2}$.*

Let $x_n = \epsilon_n + \alpha$ and $y_n = \phi_n + \alpha$ using Taylor expansion we have

$$g(y_n) = g(\phi_n + \alpha) = g(\alpha) + \phi_n g'(\alpha) + \phi_n^2 \frac{g''(\alpha)}{2} + O(\phi_n^3)$$

$$= \alpha + \phi_n g'(\alpha) + \phi_n^2 \frac{g''(\alpha)}{2} + O(\phi_n^3)$$

$$g'(x_n) = g'(\epsilon_n + \alpha) = g'(\alpha) + \epsilon_n g''(\alpha) + O(\epsilon_n^2)$$

write equation as

$$x_{n+1} = y_n - \frac{y_n - g(y_n)}{1 - g'(x_n)}$$

$$\epsilon_{n+1} + \alpha \ = \phi_n + \alpha - \frac{\epsilon_n + \alpha - (\alpha + \phi_n g'(\alpha) + \phi_n^2 \frac{g''(\alpha)}{2} + O(\phi_n^3)}{1 - (g'(\alpha) + \epsilon_n g''(\alpha) + O(\epsilon_n^2))}$$

$$\epsilon_{n+1} \ = \phi_n - \frac{\phi_n - \phi_n g'(\alpha) - \phi_n^2 \frac{g''(\alpha)}{2} - O(\phi_n^3)}{1 - g'(\alpha) - \epsilon_n g''(\alpha) - O(\epsilon_n^2))}$$

$$\epsilon_{n+1} \ = \phi_n - \frac{\phi_n - \phi_n g'(\alpha) - \phi_n^2 \frac{g''(\alpha)}{2} - O(\phi_n^3)}{1 - g'(\alpha) - \epsilon_n g''(\alpha) - O(\epsilon_n^2))}$$

$$\epsilon_{n+1} = \phi_n - \frac{\phi_n - \phi_n^2 \frac{g''(\alpha)}{2(1-g'(\alpha))} - O(\phi_n^3)}{1 - \epsilon_n \frac{g''(\alpha)}{1-g'(\alpha)} - O(\epsilon_n^2))}$$

put $\frac{g''(\alpha)}{2(1-g'(\alpha))} = C$, then

$$\epsilon_{n+1} = \phi_n - \frac{\phi_n - \phi_n^2 C - O(\phi_n^3)}{1 - 2\epsilon_n C - O(\epsilon_n^2))}$$

$$\epsilon_{n+1} = \phi_n - (\phi_n - \phi_n^2 C - O(\phi_n^3))(1 - 2\epsilon_n C - O(\epsilon_n^2))^{-1}$$

$$\epsilon_{n+1} = \phi_n - (\phi_n - \phi_n^2 C - O(\phi_n^3))(1 + 2\epsilon_n C + 4C^2\epsilon_n^2 - O(\epsilon_n^2))$$

$$\epsilon_{n+1} = \phi_n^2 C + O(\phi_n^3) - 2\epsilon_n \phi_n C + 2\epsilon_n C^2 \phi_n^2 + 2C\epsilon_n O(\phi_n^3)$$

$$-4C^2\epsilon_n^2\phi_n + 4C^3\epsilon_n^2\phi_n^2 + 4C^2\epsilon_n^2 O(\phi_n^3) - O(\phi_n^3)O(\epsilon_n^2)$$

Since ϕ_n , ϵ_n are very small, so we have

$$\epsilon_{n+1} = -2\epsilon_n \phi_n C \tag{9}$$

it can be deduce that

$$\phi_{n+1} = -2\epsilon_n \phi_{n-1} C \tag{10}$$

Now repeated substitutions in 9 and 10 gives us the following systems:

$$\epsilon_1 = -2C\epsilon_0^2$$

$$\epsilon_2 = (-2C)^3\epsilon_0^5$$

$$\epsilon_3 = (-2C)^8\epsilon_0^{13}$$

$$\epsilon_4 = (-2C)^{21}\epsilon_0^{34}$$

$$\epsilon_5 = (-2C)^{55}\epsilon_0^{89}$$

and

$$\phi_1 = (-2C)^2\epsilon_0^3$$

$$\phi_2 = (-2C)^5\epsilon_0^8$$

$$\phi_3 = (-2C)^{13}\epsilon_0^{21}$$

$$\phi_4 = (-2C)^{34}\epsilon_0^{55}$$

$$\phi_3 = (-2C)^{89}\epsilon_0^{144}$$

Note that the powers of ϵ_0 in the system are

$$2, 5, 13, 34, 89, 233, 610, 1597, \ldots,$$

and the consecutive ratios of these powers are

$$2.5, 2.6, 2.6154, 2.6177, 2.6180, 2.6180, 2.6180, \ldots \tag{11}$$

It is observable that the ratio l of two consecutive numbers in sequence (11) quickly converges to an approximate number 2.6180. The sequence (11) is related by the following relation

$$a_n = 3a_{n-1} - a_{n-2}$$

dividing by a_{n-1} and taking limit $n \longrightarrow \infty$

$$\lim_{n\longrightarrow\infty} \frac{a_n}{a_{n-1}} = 3 - \lim_{n\longrightarrow\infty} \frac{a_{n-2}}{a_{n-1}}$$

Setting $\lim_{n\longrightarrow\infty} \frac{a_n}{a_{n-1}} = \lim_{n\longrightarrow\infty} \frac{a_{n-1}}{a_{n-2}} = l$,Then $l = 3 - \frac{1}{l}$, *i.e.* $l = \frac{3+\sqrt{5}}{2} \simeq$ 2.6180.Hence Modified Algo Method has convergence order $\frac{3+\sqrt{5}}{2}$.

4 Comparison and Applications

In this section, comparison of our method with some other well-known methods is presented. "Effiency Index" is the best parameter to compare the numerical methods or finding the roots of nonlinear equations. The Fixed Point Method convergers linearly and New Iterative Method(NIM) converges quadratically and has efficiency index 1.414, But Modified Algo Method has convergence order 2.6180 and efficiency index 1.6180.

Above discussion demonstrates that Modified Algo Method converges more rapidly to the solution of a nonlinear equation. Comparison of the developed method (Modified Algo Method (MAM)) with New Iteration Method (NIM) and Fixed Point Iterative Method (FPM).

Table: Comparison of FPM, NIM, MAM				
$Method$	N	N_f	$\lvert f(x_n)\rvert$	x_n
$f(x)=\cos x - x$,				$g(x)=\cos x$, $x_0=-0.9$
FPM	80	80	$5.207054e-15$	0.7390851332151638
NIM	7	14	$6.435276e-21$	0.7390851332151606
MAM	5	10	$4.252910e-16$	0.7390851332151606
$f(x)=x^2-6x+5$, $g(x)=6-\frac{5}{x}$, $x_0=0.1$				
FPM	23	23	$7.307410e-15$	5.0000000000000018
NIM	9	18	$3.711625e-19$	1.0000000000000000
MAM	7	14	$7.475705e-24$	1.0000000000000000
$f(x)=x^2+2\ \cos(x)-3$, $g(x)=\frac{3}{x}-\frac{2\ \cos(x)}{x}$ $x_0=0.5$				
FPM	11	11	$3.440212e-17$	1.9189278163352443
NIM	6	12	$1.488864e-24$	1.9189278163352443
MAM	5	10	$7.964770e-24$	1.9189278163352443
$f(x)=x^3-4x^2+5.$, $g(x)=4-\frac{5}{x^2}$, $x_0=0.15$				
FPM	23	23	$2.146071e-14$	3.6180339887498969
NIM	11	22	$7.070459e-19$	1.3819660112501052
MAM	8	16	$9.162147e-18$	1.3819660112501052
$f(x)=\sin x-\cos x$, $g(x)=x-\sin x+\cos x$, $x_0=5$				
FPM	39	39	$4.862901e-15$	7.0685834705770334
NIM	5	10	$9.112817e-26$	3.9269908169872415
MAM	4	8	$7.168623e-25$	3.9269908169872415
$f(x)=x+2x\cos x+3\cos x-1$, $g(x)=\frac{1-3\cos x}{1+2\cos x}$, $x_0=0.8$				
FPM	33	33	$7.835191e-15$	−0.5688979502805973
NIM	14	28	$5.646651e-19$	1.6776236787874834
MAM	8	16	$2.130802e-29$	−0.5688979502805994
$f(x)=x(1+\cos x)^2-2+\sin x$, $g(x)=\frac{2-\sin x}{(1+\cos x)^2}$, $x_0=10$				
FPM	15	15	$9.537864e-16$	0.4341843066893663
NIM	8	17	$1.105793e-25$	10.4991083965501867
MAM	6	13	$1.812023e-25$	10.4991083965501867

Conclusion 5 *A modified iterative method of order 2.6180 for solving nonlinear equations is introduced. By using some examples the efficiency of the method is also discussed. The method is performing very well in comparison to the fixed point method and the new iterative method .*

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